

Non-Markovian Decoherent Quantum Walks

Peng Xue¹ and Yongsheng Zhang²

¹*Department of Physics, Southeast University, Nanjing 211189, China*

²*Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei 230026, P. R. China*

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Quantum walk acts obviously different from its classical counterpart, but decoherence will lessen and close the gap between them. To understand this process, it is necessary to investigate the evolution of quantum walk under different situation of decoherence. In this article, we study a non-Markovian decoherent quantum walk on a line. In the short time regime, the behavior of the walk deviates from both idea quantum walks and classical random walks. The position variance as a measure of quantum walk starts oscillating from the first several steps and tends to be linear on time and showing a diffusive spread in the long time limit, which is caused by the non-Markovian dephasing affecting on quantum correlations between quantum walker and his coin. We also study both quantum discord and measurement-induced disturbance as measures of quantum correlations and observe that both of them oscillate in the short time regime and tend to be zero in the long time limit. Therefore quantum walk with non-Markovian decoherence tends to diffusive spreading behavior in the long time limit, while in the short time regime it oscillates between a ballistic and diffusive spreading behavior, and the quantum correlation collapses and revivals due to the memory effect.

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I. INTRODUCTION

The quantum walk (QW) as a generalization of the random walk (RW) is important in quantum algorithm research [1, 2] because of the exponentially speed up the hitting time in glued tree graphs [3]. QWs have been demonstrated using nuclear magnetic resonance [4], trapped ions [5], single photons in bulk [6] and fibre optics [7] and the scattering of light in waveguide arrays [8–10]. With decoherence the spreading of the QW is transited from a ballistic to diffusive scaling [11, 12] and with quantumness losing the QW is transited to RW [13, 14].

An open quantum system loses its quantumness when information of the quantum state leaks into the environment around it. The unidirectional flow of information in which the decoherence and noise act consistently characterizes a Markovian process [15]. However, there are some systems such as soft- and condensed-matter systems strongly coupling to the environment and the coupling leads to a different regime where information also flows back into the system from the surroundings, which characterizes a non-Markovian process [16]. During a Markovian process, the distinguishability tends to monotonically decrease for pair of states, here in this paper, the quantum walker+coin states [11–14]. Memory effects caused by the information flowing back to the system during a non-Markovian process can temporarily increase it for the walker+coin states. In this paper, we use quantum correlation between quantum walker and his coin as a distinguishability to present the behavior of a QW on a line with a non-Markovian dephasing coin. The quantum correlation between quantum walker and his coin decays quickly but is interrupted by revivals. Correspondingly, in the short time regime, the position variance of a non-Markovian decoherent QW deviates from both those of the ideal QW and RW. In the long time limit, the quantum correlation tends to be zero and the position variance shows a diffusive spread.

This paper is organized as follows: in the next section, we introduce an idea QW on a line. In Sec. III, the non-Markovian dephasing on the coin state—two-level system is shown. We study the effects from the non-Markovian dephasing on coin. Compared to both idea QWs and RWs, non-Markovian decoherent QWs show different behavior in the short time regime. In Sec. IV, the quantum correlation between quantum walker and his non-Markovian dephasing coin is studied via the quantum discord (QD) and measurement-induced disturbance (MID). It is observed that both of QD and MID show oscillating behavior in the short time regime and tend to zero in the long time limit. In Sec. V, we analyze the behavior of non-Markovian decoherent QWs in the long time limit and observe that because of disappearance of the quantum correlation between quantum walker and his coin, a non-Markovian decoherent QW in the long time regime shows a diffusive behavior. Finally we summarize the paper briefly.

II. AN IDEAL QW ON A LINE

For an ideal QW on a line [17, 18], the Hilbert space is

$$\mathcal{H} = \mathcal{H}_w \otimes \mathcal{H}_c \quad (1)$$

with the walker Hilbert space \mathcal{H}_w spanned by the position vectors $\{|x\rangle\}$ and \mathcal{H}_c the coin space spanned by two orthogonal vectors which we denote $|\pm 1\rangle$. Each step by the walker is effected by two subsequent unitary operators: the coin-flip operator

$$C = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (2)$$

for H the Hadamard matrix and the conditional-translation operator

$$F = S \otimes |1\rangle\langle 1| + S^\dagger \otimes |-1\rangle\langle -1|, \quad (3)$$

where $S|x\rangle = |x+1\rangle$ and $S^\dagger|x\rangle = |x-1\rangle$. The resultant step operator is $U = F(\mathbb{1} \otimes C)$ with $\mathbb{1}$ the identity operator on \mathcal{H}_w .

The choice of initial state $|\psi(t=0)\rangle$ is important in studies of QWs because the interference features sensitively depend on the choice of state. This sensitivity is persistent because the dynamics are unitary hence do not die out. On the other hand the general properties of QWs do not depend on the choice of initial state so the choice of initial state is not crucial provided that the focus is on such characterization.

As we are interested in general properties, the initial state is not important so we choose the initial state with the walker at the origin of a line and holding a coin in an equal superposition of the $+1$ and -1 states:

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}} |0\rangle (|1\rangle + i|-1\rangle). \quad (4)$$

After t steps, the final state of the walker+coin system is

$$|\psi(t)\rangle = U^t |\psi(0)\rangle. \quad (5)$$

We now follow the evolution by performing a Fourier transform of the evolution operator to the “momentum” k space.

The eigenvectors

$$|k\rangle = \sum_x e^{ikx} |x\rangle, \quad (6)$$

of S and S^\dagger in Eq. (3) have the eigenrelations

$$S|k\rangle = e^{-ik} |k\rangle, S^\dagger|k\rangle = e^{ik} |k\rangle \quad (7)$$

for k a continuous real quantity. The inverse transformation is

$$|x\rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikx} |k\rangle. \quad (8)$$

The walker is initialized at the origin of a line so the walker’s initial state is

$$|0\rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |k\rangle. \quad (9)$$

In the $\{|k\rangle\}$ basis for the walker, the evolution operator becomes

$$U|k\rangle \otimes |\Phi\rangle_c = |k\rangle \otimes U_k |\Phi\rangle_c, \quad (10)$$

with $|\Phi\rangle_c$ the coin state and

$$U_k = \begin{pmatrix} e^{-ik} & e^{-ik} \\ e^{ik} & -e^{ik} \end{pmatrix}. \quad (11)$$

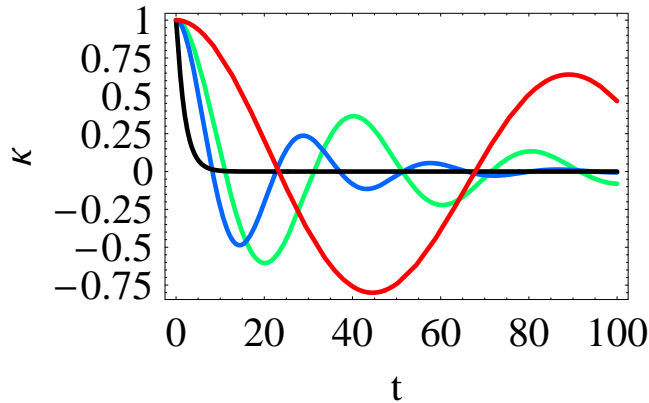


FIG. 1: (Color online) The rate κ as a function of time t with $g_0 = 1$ and different η : $\eta = 0.01$ in red, $\eta = 0.05$ in green, $\eta = 0.1$ in blue presenting non-Markovian dephasing and $\eta = 10$ in black presenting Markovian dephasing.

The general density operator for the initial state of the system in the k basis can be expressed as

$$\rho(0) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi} |k\rangle \langle k'| \otimes |\Phi_0\rangle \langle \Phi_0|. \quad (12)$$

The final state after t steps is

$$\rho(t) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dk \int_{-\pi}^{\pi} dk' |k\rangle \langle k'| \otimes U_k^t |\Phi_0\rangle \langle \Phi_0| (U_k^\dagger)^t. \quad (13)$$

In terms of the superoperator $\mathcal{L}_{kk'} \hat{O} = U_k \hat{O} U_k^\dagger$,

$$\rho(t) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dk \int_{-\pi}^{\pi} dk' |k\rangle \langle k'| \otimes \mathcal{L}_{kk'}^t |\Phi_0\rangle \langle \Phi_0|. \quad (14)$$

The walker's position is on a line labeled x with the initial position localized at 0. Measurement of the walker's position corresponds to the projection-valued measure $\{|x\rangle\langle x| \otimes \mathbb{1}; x \in \mathbb{Z}\}$ on the walker reduced state (after tracing out the coin).

The probability $P(x; t)$ that the walker will be found at the position x is

$$P(x; t) = \text{Tr} \{ |x\rangle \langle x| \otimes \mathbb{1}_c \rho(t) \}, \quad (15)$$

which is obtained by tracing over the coin of the walker+coin state and then measuring the walker's position. We can characterize $P(x; t)$ by the moments of this position distribution $\langle x^m \rangle$. The mean $\langle x \rangle$ and variance

$$\text{var} = \langle x^2 \rangle - \langle x \rangle^2 \quad (16)$$

can be used as the measure of QWs and show the signature of QWs compared to RWs. Instead of variance we can also use its square root, namely dispersion

$$\sigma = \sqrt{\text{var}} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}. \quad (17)$$

For a RW, $\sigma \sim \sqrt{t}$, which is characteristic of diffusive motion, whereas, for a QW, a quadratic enhancement is achieved: $\sigma \sim t$ [11, 12, 17, 18].

III. A QW ON A LINE WITH A NON-MARKOVIAN DEPHASING COIN IN THE SHORT TIME REGIME

We now generalize to allow for decoherence. Open quantum system loses its quantumness when information about the state leaks into its surroundings. Unidirectional flow of information characterizes a Markovian process [15].

Whereas, for a non-Markovian process [3], information also flows back into the system. Recently, though it is not well understood yet, non-Markovian processes with memory have become of central importance in the study of open system for both theory [19–22] and experiments [23–28].

Suppose that before each unitary flip of the coin, a completely positive map is performed on the coin. We consider a pure dephasing quantum process for which the density matrix $\rho_c(t) = \text{Tr}_w \rho(t)$ of the open system evolves according to the master equation

$$\frac{d}{dt} \rho_c(t) = -i \frac{\epsilon(t)}{2} [\sigma_z, \rho_c(t)] + \frac{\gamma(t)}{2} [\sigma_z \rho_c(t) \sigma_z, \rho_c(t)]. \quad (18)$$

Here $\epsilon(t)$ represents the time-dependent energy shift and $\gamma(t)$ the time-dependent rate of the decay channel described by the Pauli operator σ_z . The dephasing process influences the coherence between the walker and his coin and the evolution can be described by the time-dependent function

$$\kappa(t) = \exp \left\{ - \int_0^t dt' [\gamma(t') + i\epsilon(t')] \right\} \quad (19)$$

which is connected to the energy shift and the decay rate of the master equation (18) by the relations

$$\epsilon(t) = -\text{Im} \left[\frac{\dot{\kappa}(t)}{\kappa(t)} \right], \gamma(t) = -\text{Re} \left[\frac{\dot{\kappa}(t)}{\kappa(t)} \right]. \quad (20)$$

If the relaxation rates are positive function $\gamma(t) \geq 0$, the generator in Eq. (18) is in Lindblad form for each fixed $t \geq 0$. Such a process with $\gamma(t) \geq 0$ may be called time-dependent Markovian. The generalized master equation involving a certain memory kernel can describe a non-Markovian process, in which the rate $\gamma(t)$ must take on negative values for some interval of time. In the Markovian regime, information of the coin state leaks into its surroundings and $|\kappa(t)|$ is a monotonically decreasing function of time. In the non-Markovian environment, in contrast, information also flows back into the system of coin state and a revival of the distinguishability can be observed in the time evolution. With the definition of non-Markovianity, we see that an increase of $|\kappa(t)|$ leads to a negative rate $\gamma(t)$ in the generator of the master equation (18). As an example, we consider the case of a Lorentzian reservoir spectral density which is on resonance with the coin qubit transition frequency and leads to an exponential two point correlation function $f(t) = g_0 \eta e^{-\eta|t|}/2$, where g_0 describes the coupling strength between the coin qubit and environment and η denotes the spectral width (here we consider the coin qubit coupled to a damped Jaynes-Cummings model). The rate $\kappa(t)$ is defined as the solution of the integrodifferential equation,

$$\frac{d}{dt} \kappa(t) = - \int_0^t dt' f(t-t') \kappa(t') \quad (21)$$

corresponding to an initial condition $\kappa(0) = 1$. We obtain

$$\kappa(t) = e^{-\eta t/2} \left[\cosh \left(\frac{dt}{2} \right) + \frac{\eta}{d} \sinh \left(\frac{dt}{2} \right) \right], \quad (22)$$

where $d = \sqrt{\eta^2 - 2g_0\eta}$. We see that for weak couplings, $g_0 < \eta/2$, $|\kappa(t)|$ decreases monotonically. The correlation function $f(t)$ tends to $\delta(t)$ with η increasing. For $\eta \rightarrow \infty$, $f(t) = \delta(t)$ and the noise of different time is not correlated. Whereas, in the strong coupling regime $g_0 > \eta/2$, d turns to be a complex number and $|\kappa(t)|$ can be rewritten as

$$\kappa(t) = e^{-\eta t/2} \left[\cos \left(\frac{d't}{2} \right) + \frac{\eta}{d'} \sin \left(\frac{d't}{2} \right) \right] \quad (23)$$

with $d' = \text{Im}\{d\}$ and starts to oscillate, showing non-Markovian behavior [19–28].

The plots of $\kappa(t)$ are shown in Fig. 1. In the Markovian regime, for example $g_0 = 1$ and $\eta = 10$, the flow of information goes only from the system into the environment, and $\kappa(t)$ decreases monotonically. In the non-Markovian regime, such as $g_0 = 1$, and $\eta = 0.01, 0.05, 0.1$ respectively, memory effects however can temporarily increase it. The function $\kappa(t)$ decreases faster at $t = 0$ but will be interrupted by revivals which are due to memory effects on quantum correlations between the quantum walker and his coin.

The corresponding dynamical map which maps the coin state $\rho_c(0)$ to the state $\rho_c(t)$ at time t

$$\rho_c^{00} = \rho_c^{00}, \rho_c^{01} = \kappa^*(t) \rho_c^{01}, \rho_c^{10} = \kappa(t) \rho_c^{10}, \rho_c^{11} = \rho_c^{11}. \quad (24)$$

This map can also be written by a set of operators $\{A_1, A_2\}$ on the coin degree of freedom which satisfy

$$\sum_{n=1,2} A_n^\dagger A_n = \mathbb{1}, \quad (25)$$

where

$$\begin{aligned} A_1 &= \frac{\left(1 - \sqrt{1 - |\kappa(t)|^2}\right) |\kappa(t)|}{\kappa^*(t) \sqrt{|\kappa(t)|^2 + \left(1 - \sqrt{1 - |\kappa(t)|^2}\right)^2}} |0\rangle \langle 0| + \frac{|\kappa(t)|}{\sqrt{|\kappa(t)|^2 + \left(1 - \sqrt{1 - |\kappa(t)|^2}\right)^2}} |1\rangle \langle 1|, \\ A_2 &= \frac{|\kappa(t)|}{\sqrt{|\kappa(t)|^2 + \left(1 - \sqrt{1 - |\kappa(t)|^2}\right)^2}} |0\rangle \langle 0| + \frac{\left(1 - \sqrt{1 - |\kappa(t)|^2}\right) |\kappa(t)|}{\kappa(t) \sqrt{|\kappa(t)|^2 + \left(1 - \sqrt{1 - |\kappa(t)|^2}\right)^2}} |1\rangle \langle 1|. \end{aligned} \quad (26)$$

Considering the initial state is

$$\rho(0) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi} |k\rangle \langle k'| \otimes |\Phi_0\rangle \langle \Phi_0|. \quad (27)$$

Let the QW proceed for t steps. Then the state evolves to

$$\begin{aligned} \rho(t) &= \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi} |k\rangle \langle k'| \otimes \sum_{n_1, \dots, n_t} U_k A_{n_t} \cdots U_k A_{n_1} |\Phi_0\rangle \langle \Phi_0| A_{n_1}^\dagger U_k^\dagger \cdots A_{n_t}^\dagger U_k^\dagger \\ &= \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi} |k\rangle \langle k'| \otimes \mathcal{L}_{kk'}^t |\Phi_0\rangle \langle \Phi_0|. \end{aligned} \quad (28)$$

Note that for $k = k'$ this superoperator preserves the trace. This implies that

$$\text{Tr} \left(\mathcal{L}_{kk}^t \hat{O} \right) = \text{Tr} \left(\hat{O} \right) \quad (29)$$

for any operator \hat{O} .

Now we consider the effect of the non-Markovian dephasing on coin. As we know, without decoherence the variance of position distribution of QW is quadratically dependent on time t . With Markovian dephasing on coin, in the long time limit the transition from a ballistic to a diffusion spreading behavior is observed. In this paper, we will show the behavior of a QW with non-Markovian dephasing coin. The variance is used as a signature of the QW compared to RW. In the short time regime we can calculate the variance of position distribution numerically. In Fig. 2 from the numerical results we can see the variance as a function of time starts to oscillate from the first several steps and the period of oscillations depends on the coefficient d . Compared to that of ideal QW, in the short time regime ($t \lesssim 100$) the variance of the QW with non-Markovian dephasing coin deviates from a ballistic spreading behavior. However, in comparison with RW, the oscillating variance neither shows a diffusion behavior, which is due to effect on quantum correlations between quantum walker and his coin from the non-Markovian decoherence. From the numerical calculations the position variance of a RW distribution is the lower bound of that of QW with non-Markovian dephasing coin.

In Fig. 3, we plot position distribution for a QW with a non-Markovian dephasing coin ($g_0 = 1$ and $\eta = 0.01$) at the 38th, 42nd, 46th, and 50th steps. At the 38th step, the position distribution shows RW behavior, and at the 42nd and 46th steps, there are two small peaks higher than the others nearby in the position distribution. It shows the memory effect from non-Markovian decoherence draws QW behavior back a little. The small peaks disappear at the 50th step. The plots of position distribution show the behavior of QW with non-Markovian dephasing coin oscillates between the quantum (ballistic) and classical (diffusive) behaviors.

IV. QUANTUM CORRELATIONS DYNAMICS IN NON-MARKOVIAN ENVIRONMENT

The signature of a decoherent discrete-time QW on a line can be demonstrated in various ways such as the position probability distribution becoming increasingly Gaussian, with a concomitant fall in the standard deviation. Both of them do not recover the correlations between the quantum walker and his coin, which is known as the reason causing

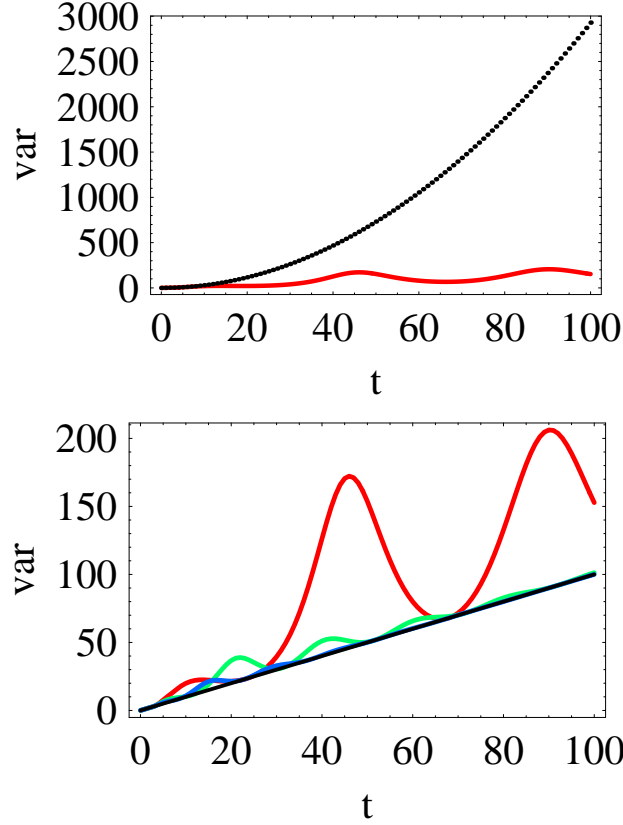


FIG. 2: (Color online) The variances for a QW on a line after $t = 100$ steps with a perfect coin in dotted-line, with a non-Markovian dephasing coin with parameters $g_0 = 1$ and $\eta = 0.01$ in red and $\eta = 0.05$ in green, $\eta = 0.1$ in blue and $\eta = 10$ in black. Here we choose $(|0\rangle + i|1\rangle)/\sqrt{2}$ as an initial coin state for calculations. (a) Comparison of variances between the RW and QW with a non-Markovian dephasing coin with parameters $g_0 = 1$ and $\eta = 0.01$. (b) Comparison of variances between the QW with a non-Markovian dephasing coin with $g_0 = 1$ and $\eta = 0.01$, $\eta = 0.05$, $\eta = 0.1$ and $\eta = 10$.

the quantum behavior of QWs. In this section we study the quantum correlation of a decoherent QW by two popular measures, measurement-induced disturbance (MID) [29–31] and quantum discord (QD) [32, 33].

Given the walker+coin state $\rho(t)$, let the density matrix for the system be diagonalized to

$$\rho_i = \sum_j p_i^j \Pi_i^j, \quad (30)$$

for $i = w, c$, where $\{\Pi_i^j\}$ is a complete projection-valued measure (i.e. using von Neumann measurements) for walker w or coin c . Summing over joint projections on the state yields the diagonalized state

$$\Pi\rho(t) = \sum_k \mathbb{1}_w \otimes \Pi_c^k \rho(t) \mathbb{1}_w \otimes \Pi_c^k. \quad (31)$$

The leftmost Π is an operator on the density matrix that diagonalizes it in the spectral basis corresponding to Π_i^j projective measurements. The operator Π can also be described as a ‘local measurement strategy’.

Correlations between the walker ρ_w and coin state ρ_c are regarded as classical if there is a unique local measurement strategy Π leaving $\Pi\rho(t)$ unaltered from the original walker+coin state $\rho(t)$ [29–31]. We ascertain whether the walker+coin state is ‘quantum’ by determining whether a local measurement strategy exists that leaves the state unchanged.

The degree of quantumness is given by the MID [29–31]

$$Q[\rho(t)] = I[\rho(t)] - I[\Pi\rho(t)], \quad (32)$$

for $I[\rho(t)] = S[\rho_w(t)] + S[\rho_c(t)] - S[\rho(t)]$ the quantum mutual information and $S[\rho(t)] = -\text{Tr}[\rho(t) \log_2 \rho(t)]$ denotes von Neumann entropy. By construction $\Pi\rho(t)$ is classical. Thus, the MID is the difference between the quantum and

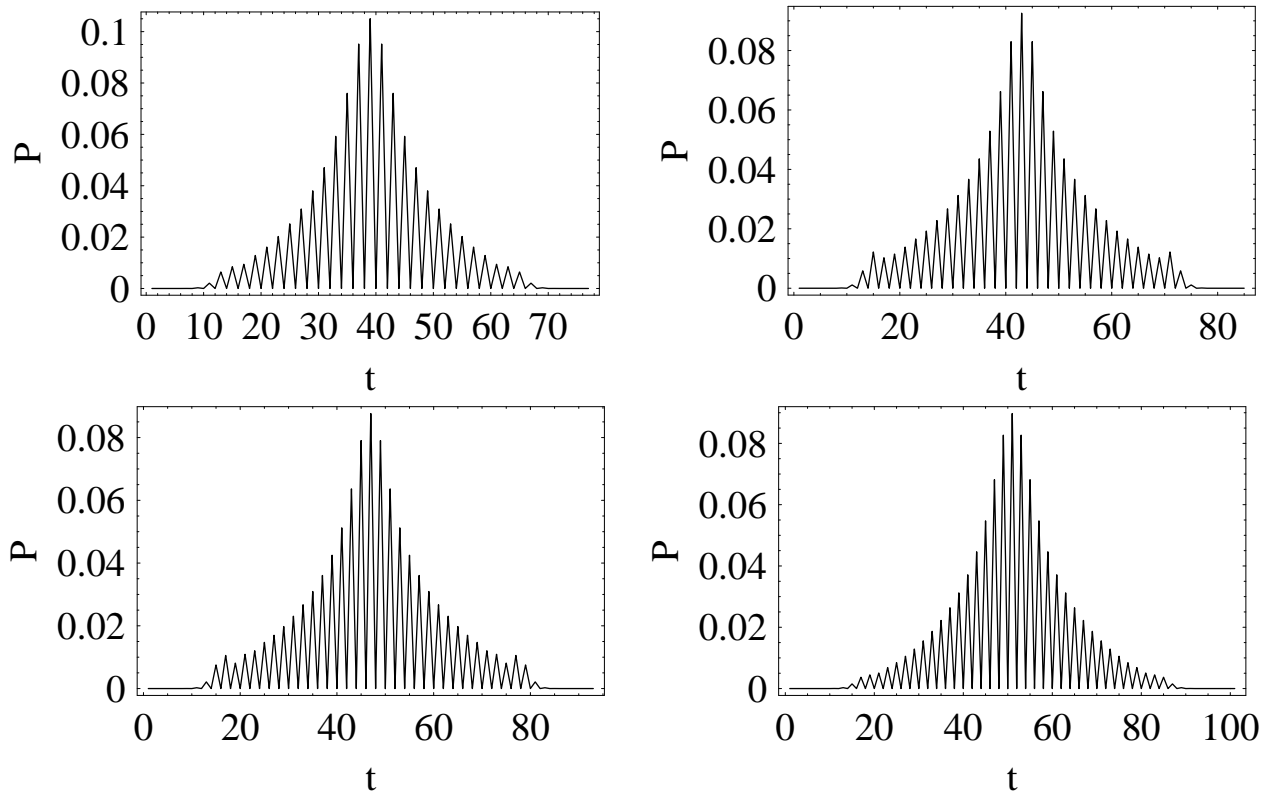


FIG. 3: (Color online) The position distribution for a QW on a line with a non-Markovian dephasing coin with parameters $g_0 = 1$ and $\eta = 0.01$ at the 38th step (a), the 42nd step (b), the 46th step (c) and the 50th step (d).

classical mutual information, which quantify total and classical correlations. Accordingly, Eq. (32) is interpreted as the difference between the total and classical correlations, which are represented by the quantum mutual information and the mutual information.

QD [32, 33] is defined by

$$D[\rho(t)] = I[\rho(t)] - \sup_{\Pi} I[\Pi\rho(t)]. \quad (33)$$

The MID has an operational definition, unlike QD, it also tends to overestimate non-classicality because of lack of optimization over local measurements. Applied to QWs, we find that MID, while acting as a loose upper bound on QD, still tends to reflect well trends in the behavior of the latter.

We numerically calculate the MID and QD for a QW with perfect, Markovian and non-Markovian dephasing coins in the short time regime. The quantum correlation between quantum walker and his coin presented by the MID and QD is oscillating for the first several steps ($\lesssim 100$) shown in Fig. 4(a) and (b). Without decoherence there is no difference between MID and QD. Memory effects caused by the non-Markovianity of the environment deviate the MID and QD from monotonically behavior as that during the Markovian process. Both MID and QD decay steeply and interrupted by revivals. The period of oscillating is as same as that of corresponding position variance, which proves that quantum correlation between quantum walker and his coin results in the behavior of the QW. Either of the MID and QD tends to zero with time increasing. Whereas for ideal QWs, both MID and QD do not decrease with time as shown in [34, 35]. From the numerical results one can see the quantum correlation in the case of Markovian decoherence is the lower bound of that with Non-Markovian decoherence, which explains the reason that the position variance of a RW distribution provides the lower bound of that of QW with non-Markovian dephasing coin. In comparison of MID and QD shown in Fig. 4(c), both of them can show the quantum correlation between quantum walker and his coin changing with time and MID oscillates with bigger amplitude than QD, which makes that the MID might be better signature for observing in this case.

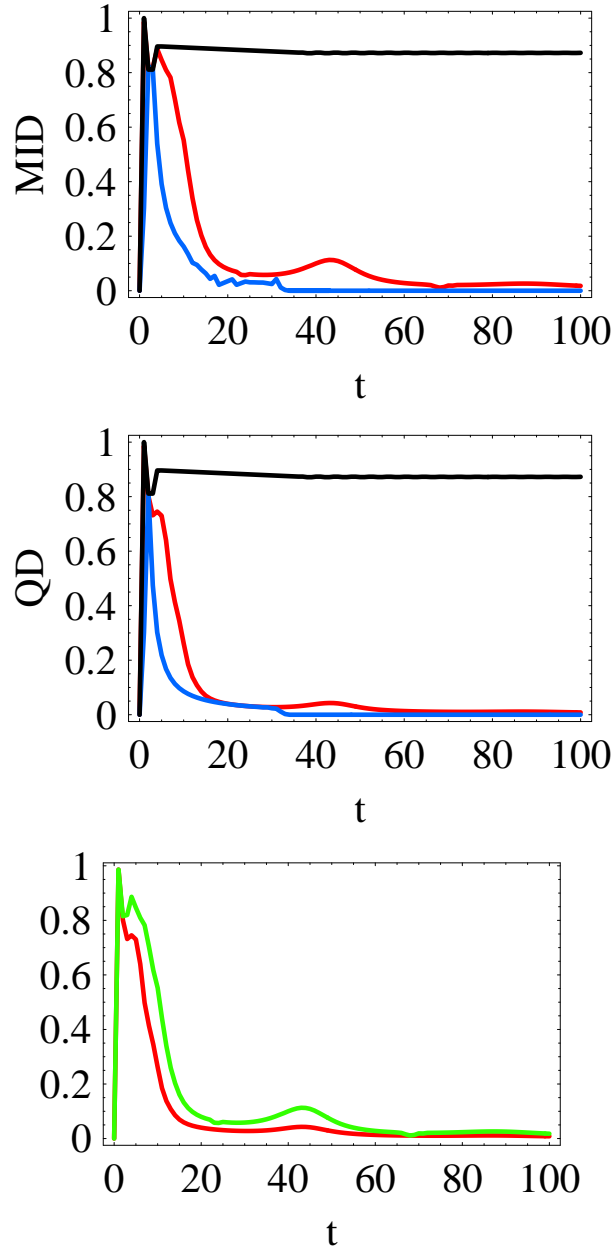


FIG. 4: (Color online) (a) The MID, (b) QD for a QW on a line after $t = 100$ steps with perfect (in black), non-Markovian dephasing coin (in red) and Markovian dephasing coin (in blue), respectively. Here we choose the parameters $g_0 = 1$, $\eta = 0.01$ and $\eta = 10$ respectively for calculations. (c) The comparison of MID in red and QD in green for a QW on a line with the same parameters above.

V. A QW ON A LINE WITH A NON-MARKOVIAN DEPHASING COIN IN THE LONG TIME LIMIT

In the above sections we show the behavior of QW on a line with a non-Markovian dephasing coin results in the quantum correlation between quantum walker and his coin, which is presented by the MID and QD. We observe that both MID and QD decrease with time and tend to zero in the long time limit. That means the quantum correlation will disappear for long enough time. In this section, we will show in the long time regime, the behavior of a non-Markovian decoherent QW.

The probability for the walker to reach a point x at time t is

$$\begin{aligned} P(x; t) &= \text{Tr}\{|x\rangle \langle x| \rho(t)\} \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dk \int_{-\pi}^{\pi} dk' e^{-i(k-k')x} \text{Tr}\{\mathcal{L}_{kk'}^t |\Phi_0\rangle \langle \Phi_0|\}. \end{aligned} \quad (34)$$

Thus we can calculate the moments of this distribution:

$$\begin{aligned} \langle x^m \rangle &= \sum_x x^m P(x; t) \\ &= \frac{(-1)^m}{(2\pi)^2} \int_{-\pi}^{\pi} dk \int_{-\pi}^{\pi} dk' \delta^{(m)}(k - k') \text{Tr}\{\mathcal{L}_{kk'}^t |\Phi_0\rangle \langle \Phi_0|\}. \end{aligned} \quad (35)$$

We can then invert the order of operations and do the x sum first. This sum can exactly be carried out in terms of derivatives of the δ function:

$$\frac{1}{2\pi} \sum_x x^m e^{-ix(k-k')} = (-i)^m \delta^{(m)}(k - k'). \quad (36)$$

Inserting this result back into the expression for $\langle x \rangle$ we get the first moment

$$\begin{aligned} \langle x \rangle &= \frac{-i}{2\pi} \int_{-\pi}^{\pi} dk \frac{d}{dk} \text{Tr}\{\mathcal{L}_k^t |\Phi_0\rangle \langle \Phi_0|\} \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \sum_{j=1}^t \text{Tr}\{\sigma_z \mathcal{L}_k^j |\Phi_0\rangle \langle \Phi_0|\}. \end{aligned} \quad (37)$$

We can carry out a similar integration by parts to get the second moment:

$$\langle x^2 \rangle = -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \left\{ \sum_{j=1}^t \sum_{j'=1}^j \text{Tr}\left\{ \sigma_z \mathcal{L}_k^{j-j'} (\sigma_z \mathcal{L}_k^{j'} |\Phi_0\rangle \langle \Phi_0|) \right\} + \sum_{j=1}^t \sum_{j'=1}^{j-1} \text{Tr}\left\{ \sigma_z \mathcal{L}_{kj}^{j-j'} \left[(\mathcal{L}_k^{j'} |\Phi_0\rangle \langle \Phi_0|) \sigma_z \right] \right\} \right\}. \quad (38)$$

Because \mathcal{L}_k is a linear transformation we can represent it as a matrix acting on the space of 2×2 operators. We choose the representation as

$$\hat{O} = r_1 \mathbb{1} + r_2 \sigma_x + r_3 \sigma_y + r_4 \sigma_z. \quad (39)$$

The action of \mathcal{L}_k on \hat{O} is given by the matrix

$$\mathcal{L}_k \hat{O} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin 2k \text{Im}[\kappa(t)] & \sin 2k \text{Re}[\kappa(t)] & \cos 2k \\ 0 & -\cos 2k \text{Im}[\kappa(t)] & -\cos 2k \text{Re}[\kappa(t)] & \sin 2k \\ 0 & \text{Re}[\kappa(t)] & -\text{Im}[\kappa(t)] & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} \quad (40)$$

Because $r_1 = \text{Tr}(\hat{O}) = \text{Tr}(\mathcal{L}_k \hat{O})$. The only nontrivial dynamics results from the 3×3 submatrix

$$M_k = \begin{pmatrix} 0 & 0 & 0 \\ \sin 2k \text{Im}[\kappa(t)] & \sin 2k \text{Re}[\kappa(t)] & \cos 2k \\ -\cos 2k \text{Im}[\kappa(t)] & -\cos 2k \text{Re}[\kappa(t)] & \sin 2k \\ \text{Re}[\kappa(t)] & -\text{Im}[\kappa(t)] & 0 \end{pmatrix} \quad (41)$$

We also need to know the effects of left and right multiplying by σ_z . These are given by the two matrices

$$Z_L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, Z_R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (42)$$

Let us take these expressions and apply them to Eq. (37) for the first moment. In the integrand, the initial density matrix for the coin is multiplied j times by \mathcal{L}_k , then left multiplied by σ_z and finally the trace is taken. Given the

above expression for Z_L , we see that this is the same as multiplying the 3-vector (r_2, r_3, r_4) j times by M_k and then keeping only the r_4 component of the result. This gives us the new expression for the first moment

$$\langle x \rangle = -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \left(\sum_{j=1}^t M_k^j \right) \begin{pmatrix} r_2 \\ r_3 \\ r_4 \end{pmatrix} \quad (43)$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \left[(1 - M_k)^{-1} (M_k - M_k^{t+1}) \right] \begin{pmatrix} r_2 \\ r_3 \\ r_4 \end{pmatrix}. \quad (44)$$

The eigenvalues of M_k satisfy $0 < |\lambda| < 1$ and then in the long time limit, therefore $M_k^{t+1} \rightarrow 0$ and the first moment becomes approximately

$$\langle x \rangle \approx -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \left[(1 - M_k)^{-1} M_k \right] \begin{pmatrix} r_2 \\ r_3 \\ r_4 \end{pmatrix} \quad (45)$$

$$= \frac{r_2 \text{Re}[\kappa(t)] - r_3 \text{Im}[\kappa(t)] + r_4 |\kappa(t)|^2}{-1 + |\kappa(t)|^2}. \quad (46)$$

In the long time limit, the second moment can also be calculated as

$$\begin{aligned} \langle x^2 \rangle &= t - \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \left[Z_L \sum_{j=1}^t \sum_{j'=1}^{j-1} \mathcal{L}_k^{j-j'} (Z_L + Z_R) \mathcal{L}_k^{j'} \right] \begin{pmatrix} 1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} \\ &= t - \sum_{j=1}^t \sum_{j'=1}^{j-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} M_k^{j-j'} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \\ &= t - \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \left[t - (1 - M_k)^{-1} M_k + (1 - M_k)^{-1} M_k^t \right] (1 - M_k)^{-1} M_k \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \\ &= t \left(1 + \frac{2|\kappa(t)|^2}{-1 + |\kappa(t)|^2} \right) + \frac{2|\kappa(t)|^4 + 5|\kappa(t)|^2}{(-1 + |\kappa(t)|^2)^2}, \end{aligned} \quad (47)$$

which is independent on the initial coin state.

The position variance is obtained as

$$\text{var} = \langle x^2 \rangle - \langle x \rangle^2 = t \left(1 + \frac{2|\kappa(t)|^2}{-1 + |\kappa(t)|^2} \right) + \frac{2|\kappa(t)|^4 + 5|\kappa(t)|^2}{(-1 + |\kappa(t)|^2)^2} - \frac{r_2 \text{Re}[\kappa(t)] - r_3 \text{Im}[\kappa(t)] + r_4 |\kappa(t)|^2}{-1 + |\kappa(t)|^2}. \quad (48)$$

In the long time regime, $\kappa(t)$ (22) tends to zero. Then the position variance is linear dependent on time t . Thus in the case of the non-Markovian dephasing coin the variance grows linearly with time and a QW with a non-Markovian dephasing coin shows a diffusive spread in the long time limit.

In the long time limit, we can obtain the analytical results of the position variance of a non-Markovian decoherent QW. In Fig. 5, we can see that for the first several steps, the numerical and analytical results agree with each other. Whereas for long time, the variance becomes linear dependent on time t showing a diffusive spread, just as in the classical case, though the rate of growth will greater than for the RW.

VI. CONCLUSION

In this paper, we presented how non-Markovian decoherence can influence the evolution of a quantum walker on a line and showed a different behavior compared to the ballistic and diffusive spreads. We analyze the dynamics and quantum correlation of quantum walker and his coin. We use the MID and QD as measures for quantum correlations. The QD would be a valuable measure to use and the MID also suffices to show that pure quantum correlation induced by having non-Markovian dephasing on the coin state. The memory effect caused by the information flowing back to the system during a non-Markovian process decreases the quantum correlation quickly which is later interrupted

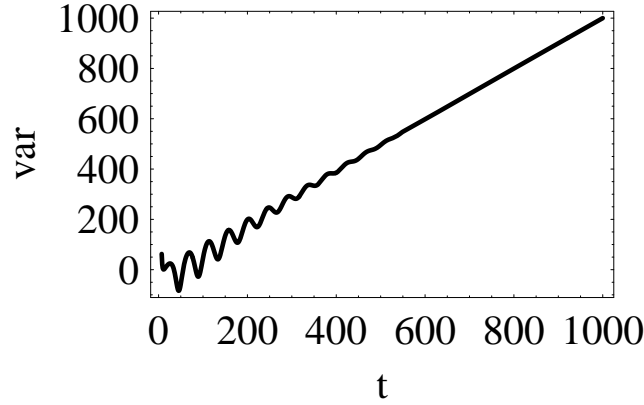


FIG. 5: The analytical result on the variance of position distribution for a QW on a line with a non-Markovian dephasing coin in the long time regime. Here we choose the parameters $g_0 = 1$ and $\eta = 0.01$ for calculations.

by revivals. In the long time limit, the quantum correlation disappears with $\kappa(t)$ tending to zero, which results in a transition from a QW with a ballistic scaling to a RW with a diffusive scaling. We draw the conclusions as followings: in the short time the behavior of QW with non-Markovian dephasing coin oscillates between a ballistic and diffusive spreading behavior, and the quantum correlation between the walker and coin collapses and revivals due to the memory effect. Whereas in the long time limit, QW with non-Markovian decoherence does not make much difference from that with Markovian decoherence. Thus our analysis is quite valuable in that we characterize this QW on a line with non-Markovian dephasing coin carefully and devise appropriate, meaningful QD and MID as measures to study quantum correlation dynamics for this system.

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